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Approximation by Generalized Faber Series in Bergman Spaces on Finite Regions with a Quasiconformal Boundary

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In this work, for the first time, generalized Faber series for functions in the Bergman space $A^2(G)$ on finite regions with a quasiconformal boundary are defined, and their convergence on compact subsets of G and with respect to the norm on $A^2(G)$ is investigated. Finally, if $S_n(f, z)$ is the nth partial sum of the generalized Faber series of $f \in A^2(G)$, the discrepancy $\|f - S_n(f, \cdot)\|_{A^2(G)}$ is evaluated by $E_n(f, G)$, the best approximation to f by polynomials of degree n.

1. INTRODUCTION AND STATEMENT OF PROBLEM

Let G be a finite region with $0 \in G$ bounded by a quasiconformal curve Γ. We recall that Γ is called a quasiconformal curve if there exists a quasiconformal homeomorphism of the complex plane onto itself that maps a circle onto Γ. The set

$$A^2(G) := \left\{ f : f \text{ is analytic in } G \text{ and } \iint_G |f(z)|^2 \, d\sigma_z < +\infty \right\}$$

is called the Bergman space on G, where $d\sigma_z$ denotes the Lebesque measure in the complex plane. It is well known that $A^2(G)$ is a Hilbert space with the inner product

$$\langle f, g \rangle := \iint_G f(z) \, \overline{g(z)} \, d\sigma_z,$$

and the set of polynomials are dense in $A^2(G)$ with respect to the norm $\|f\|_{A^2(G)} := (\langle f, f \rangle)^{1/2}$, see [8: Ch. I] and [5: p. 420]. Also, for $n = 1, 2, ...,$

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if the degree of the best approximation to f by polynomials of degree $\leqslant n$ is defined by the formula

$$E_n(f, G) := \operatorname{Inf}\{ \|f - P\|_{A^2(G)} : P(z) \text{ is a polynomial of degree } \leqslant n\},$$

there exists a polynomial $P_n^*(z)$, of degree $\leqslant n$, such that $E_n(f, G) := \|f - P_n^*\|_{A^2(G)}$. $P_n^*(z)$ is called the best approximant polynomial to $f \in A^2(G)$.

Let U be the open unit disc and $w = \varphi(z)$ the conformal mapping of $C\bar{G}$ onto $C\bar{U}$ with normalization $\varphi(\infty) = \infty$ and $\lim_{z \to \infty} (1/z)\, \varphi(z) > 0$, where $C\bar{G}$ and $C\bar{U}$ are the complements of $\bar{G}$ and $\bar{U}$ in the extended complex plane respectively. We denote the inverse of $\varphi(z)$ by $\Psi(w)$. If $F_m(z)$ is the mth Faber polynomial for $\bar{G}$, then it is known that for every $(z, w) \in \bar{G} \times C\bar{U}$

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{m=0}^{\infty} \frac{F_m(z)}{w^{m+1}}$$

and the series converges uniformly and absolutely on compact subsets of $\bar{G} \times C\bar{U}$. From this, we get

$$\frac{\Psi'(w)}{(\Psi(w) - z)^2} = \sum_{m=1}^{\infty} \frac{F'_m(z)}{w^{m+1}}, \qquad (z, w) \in \bar{G} \times C\bar{U}, \tag{1}$$

where the series converges absolutely and uniformly on compact subsets of $\bar{G} \times C\bar{U}$. More information for Faber polynomials and Faber expansions can be found in [8] and [11].

In this work, we define a generalized Faber series of a function $f \in A^2(G)$ to be of the form $\sum_{m=1}^{\infty} a_m(f)\, F'_m(z)$, and we show that it converges uniformly on compact subsets of G. Furthermore, if $S_n(f, z) := \sum_{m=1}^{n+1} a_m(f)\, F'_m(z)$ is the nth partial sum of a generalized Faber series of $f \in A^2(G)$, then we give an estimation of the discrepancy $\|f - S_n(f, \cdot)\|_{A^2(G)}$ by means of $E_n(f, G)$. Also, if a series $\sum_{m=1}^{\infty} a_m F'_m(z)$ is convergent to $f \in A^2(G)$ with respect to the norm $\|\ \|_{A^2(G)}$, we show that the a_m are the generalized Faber coefficients $a_m(f)$ of f. In [9], similar problems were studied by D. M. Israfilov in $A(\bar{G})$, where $A(\bar{G})$ denotes the class of functions which are analytic in G and continuous in $\bar{G}$.

2. DEFINITIONS AND SOME AUXILIARY RESULTS

In [4], V. I. Belyi gave the following integral representation for the functions $f \in A(\bar{G})$ as

$$f(z) = -\frac{1}{\pi} \iint_{C\bar{G}} \frac{(f \circ y)(\zeta)}{(\zeta - z)^2}\, y_{\bar{\zeta}}(\zeta)\, d\sigma_\zeta, \qquad z \in G, \tag{2}$$

and he studied the approximation by polynomials in $A(\bar{G})$. Here $y(z)$ is a K-quasiconformal reflection with respect to Γ, i.e., a sense-reversing K-quasiconformal involution of the extended complex plane taking G into $C\bar{G}$ and keeping every point of Γ fixed, with $y(0)=\infty$ and $y(\infty)=0$. Such a mapping of the plane does exist [10]. It follows from Ahlfors' lemma [1: p. 80] that the reflection $y(z)$ may always be chosen to be differentiable everywhere, except possibly on the points of $\Gamma \cup \{0\}$. In this case, I. M. Batchaev generalized the integral representation above to Lebesque integrable and analytic functions, and so to functions $f \in A^2(G)$ [3]. The analog of the integral representation (2) for unbounded domains with boundary passing through ∞ was first proved by L. Bers [6].

Quasiconformal mappings and quasiconformal reflections with respect to Γ are examined in [1] and [10] in great detail. From now on, the reflection $y(z)$ will be a differentiable K-quasiconformal reflection with respect to Γ.

Let $f \in A^2(G)$. Substituting $\zeta = \Psi(w)$ in (2), we get

$$f(z) = -\frac{1}{\pi} \iint_{C\bar{U}} f(y(\Psi(w)))\, \overline{\Psi'}(w)\, y_{\bar{\zeta}}(\Psi(w))\, \frac{\Psi'(w)}{(\Psi(w)-z)^2}\, d\sigma_w, \quad z \in G, \quad (3)$$

Thus, if we consider (1) and (3), and define coefficients $a_m(f)$, by

$$a_m(f) := -\frac{1}{\pi} \iint_{C\bar{U}} \frac{f(y(\Psi(w)))\, \overline{\Psi'}(w)}{w^{m+1}}\, y_{\bar{\zeta}}(\Psi(w))\, d\sigma, \qquad m = 1, 2, \ldots \quad (4)$$

then we can associate a formal series $\sum_{m=1}^{\infty} a_m(f)\, F'_m(z)$ with the function $f \in A^2(G)$, i.e.,

$$f(z) \sim \sum_{m=1}^{\infty} a_m(f)\, F'_m(z). \quad (5)$$

We call this formal series a generalized Faber series of $f \in A^2(G)$, and the coefficients $a_m(f)$ are called generalized Faber coefficients of f.

LEMMA 2.1. *Let $\{F_m(z)\}$ be the Faber polynomials for $\bar{G}$. Then*

$$\sum_{m=1}^{n} \frac{\|F'_m\|^2_{A^2(G)}}{m} \leqslant n\pi.$$

Proof. Let $S_m(G)$ be the area of the image of G under F_m in the Riemann surface of F_m. Since $F_m(\Psi(w)) = w^m + \sum_{v=1}^{\infty} m b_{mv} w^{-v}$, $|w| > 1$, [7: p. 43] where the b_{mv} are the Grunsky coefficients, by means of a theorem due to N. A. Lebedev and I. M. Millin in [11: p. 213] we have

$$S_m(G) = \pi\left(m - \sum_{v=1}^{\infty} vm^2\,|b_{mv}|^2\right) \leqslant m\pi. \tag{6}$$

On the other hand

$$S_m(G) = \iint_G |F'_m(z)|^2\, d\sigma_z = \|F'_m\|^2_{A^2(G)}. \tag{7}$$

From (6) and (7), it follows that

$$\sum_{m=1}^{n} \frac{\|F'_m\|^2_{A^2(G)}}{m} \leqslant n\pi.$$

Let us emphasize that we can, in general, not reduce $n\pi$ in the inequality above. In fact, if we consider the unit disc U, then $F_m(z) = z^m$ and $\sum_{m=1}^{n} \|F'_m\|^2_{A^2(U)}/m = n\pi$.

Remark. Using a completely similar method, it can be proved that this lemma is true for any continuum E whose complement is connected.

LEMMA 2.2. *The series* $\sum_{m=1}^{\infty} |F'_m(z)|^2/(m+1)$ *is convergent uniformly on compact subsets of* G.

Proof. Let z be a fixed point in G. Then the power series $\sum_{m=1}^{\infty} (F'_m(z)/(m+1))\, w^{m+1}$ defines an analytic function $A(z, w)$ in U, i.e.,

$$A(z, w) := \sum_{m=1}^{\infty} \frac{F'_m(z)}{m+1}\, w^{m+1}, \qquad w \in U, \tag{8}$$

Thus, by taking derivative of (8) with respect to w and considering (1) we get

$$A'(z, w) = \sum_{m=1}^{\infty} F'_m(z)\, w^m = \frac{\Psi'(1/w)}{(\Psi(1/w) - z)^2\, w}, \qquad w \in U, \tag{9}$$

Let $0 < r < 1$. Since $\sum_{m=1}^{\infty} F'_m(z)\, w^m$ is convergent uniformly and absolutely on the closed disc $\bar{D}(0, r)$, it follows that

$$\iint_{\bar{D}(0,r)} |A'(z, w)|^2\, d\sigma_w = \pi \sum_{m=1}^{\infty} \frac{|F'_m(z)|^2}{m+1}\, r^{2m+2}. \tag{10}$$

(9) and (10) show that

$$\pi \sum_{m=1}^{\infty} \frac{|F_m'(z)|^2}{m+1} r^{2m+2} = \iint_{\bar{D}(0,r)} \left| \frac{\Psi'(1/w)}{(\Psi(1/w)-z)^2 w} \right|^2 d\sigma_w. \tag{11}$$

Since

$$\lim_{r\to 1^-} \iint_{\bar{D}(0,r)} \left| \frac{\Psi'(1/w)}{(\Psi(1/w)-z)^2 w} \right|^2 d\sigma_w = \iint_U \left| \frac{\Psi'(1/w)}{(\Psi(1/w)-z)^2 w} \right|^2 d\sigma_w,$$

and

$$\iint_U \left| \frac{\Psi'(1/w)}{(\Psi(1/w)-z)^2 w} \right|^2 d\sigma_w < +\infty$$

we get

$$\pi \sum_{m=1}^{\infty} \frac{|F_m'(z)|^2}{m+1} = \iint_U \left| \frac{\Psi'(1/w)}{(\Psi(1/w)-z)^2 w} \right|^2 d\sigma_w.$$

On the other hand, it can be easily proved that

$$\iint_U \left| \frac{\Psi'(1/w)}{(\Psi(1/w)-z)^2 w} \right|^2 d\sigma_w$$

is continuous in G. So, by Dini's theorem the series $\sum_{m=1}^{\infty} |F_m'(z)|^2/(m+1)$ is convergent uniformly on compact subsets of G.

LEMMA 2.3. *If $f \in A^2(G)$ and $y(\zeta)$ is a differentiable* K-*quasiconformal reflection with respect to Γ, then*

$$\iint_{C\bar{G}} |(f\circ y)(\zeta)|^2 \, |y_{\bar{\zeta}}(\zeta)|^2 \, d\sigma_\zeta \leqslant \frac{\|f\|^2_{A^2(G)}}{1-k^2},$$

where $k := (K-1)/(K+1)$.

Proof. Since $\bar{y}(\zeta)$ is a differentiable K-quasiconformal mapping of the extended complex plane onto itself, we have $|\bar{y}_{\bar{\zeta}}|/|\bar{y}_\zeta| \leqslant k$ and $|\bar{y}_\zeta|^2 - |\bar{y}_{\bar{\zeta}}|^2 > 0$. Also, it is known that $|\bar{y}_{\bar{\zeta}}| = |y_\zeta|$ and $|\bar{y}_\zeta| = |y_{\bar{\zeta}}|$. Therefore, since $|y_\zeta|/|y_{\bar{\zeta}}| \leqslant k$ and $|y_{\bar{\zeta}}|^2 - |y_\zeta|^2 > 0$, we get

$$\iint_{C\bar{G}} |(f\circ y)(\zeta)|^2\,|y_{\bar{\zeta}}(\zeta)|^2\,d\sigma_\zeta = \iint_{C\bar{G}} |(f\circ y)(\zeta)|^2\,(1-(|y_\zeta|/|y_{\bar{\zeta}}|)^2)^{-1} \times (|y_{\bar{\zeta}}|^2 - |y_\zeta|^2)\,d\sigma_\zeta$$

$$\iint_{C\bar{G}} |(f\circ y)(\zeta)|^2\,|y_{\bar{\zeta}}(\zeta)|^2\,d\sigma_\zeta \leqslant \frac{1}{1-k^2}\iint_{C\bar{G}} |(f\circ y)(\zeta)|^2\,(|y_{\bar{\zeta}}|^2 - |y_\zeta|^2)\,d\sigma_\zeta.$$

Since the Jacobian of $y(\zeta)$ is $(|y_\zeta|^2 - |y_{\bar{\zeta}}|^2)$, if we substitute z for $y(\zeta)$ on the right side of the inequality above we get

$$\iint_{C\bar{G}} |(f\circ y)(\zeta)|^2\,|y_{\bar{\zeta}}(\zeta)|^2\,d\sigma_\zeta \leqslant \frac{\|f\|^2_{A^2(G)}}{1-k^2}.$$

3. MAIN RESULTS

THEOREM 3.1. *Let $f\in A^2(G)$. If $\sum_{m=1}^{\infty} a_m(f)\,F'_m(z)$ is a generalized Faber series of f, then the series $\sum_{m=1}^{\infty} a_m(f)\,F'_m(z)$ converges uniformly to f on compact subsets of G.*

Proof. Let M be a compact subset of G and $y(z)$ a differentiable K-quasiconformal reflection with respect to Γ. Since for $z\in M$

$$f(z) = -\frac{1}{\pi}\iint_{C\bar{G}} \frac{(f\circ y)(\zeta)}{(\zeta-z)^2}\,y_{\bar{\zeta}}(\zeta)\,d\sigma_\zeta$$
$$= -\frac{1}{\pi}\iint_{C\bar{U}} f(y(\Psi(w)))\,\overline{\Psi'}(w)\,y_{\bar{\zeta}}(\Psi(w))\,\frac{\Psi'(w)}{(\Psi(w)-z)^2}\,d\sigma_w$$

and

$$a_m(f) = -\frac{1}{\pi}\iint_{C\bar{U}} \frac{f(y(\Psi(w)))\,\overline{\Psi'(w)}}{w^{m+1}}\,y_{\bar{\zeta}}(\Psi(w))\,d\sigma_w, \qquad m=1, 2, ...$$

we obtain by means of Hölder's inequality and Lemma 2.3

$$\left| f(z) - \sum_{m=1}^{n} a_m(f)\,F'_m(z)\right| \leqslant \frac{\|f\|_{A^2(G)}}{\pi\sqrt{1-k^2}}\left(\iint_{C\bar{U}} \left|\frac{\Psi'(w)}{(\Psi(w)-z)^2} - \sum_{m=1}^{n}\frac{F'_m(z)}{w^{m+1}}\right|^2 d\sigma_w\right)^{1/2} \tag{12}$$

for every $z\in M$.

Let $1 < r < R < +\infty$. In view of (1)

$$\iint_{r<|w|<R} \left| \frac{\Psi'(w)}{(\Psi(w)-z)^2} - \sum_{m=1}^{n} \frac{F'_m(z)}{w^{m+1}} \right|^2 d\sigma_w = \iint_{r<|w|<R} \left| \sum_{m=n+1}^{\infty} \frac{F'_m(z)}{w^{m+1}} \right|^2 d\sigma_w$$

$$= \pi \sum_{m=n+1}^{\infty} \frac{1}{m} \left(\frac{1}{r^{2m}} - \frac{1}{R^{2m}} \right) |F'_m(z)|^2$$

$$\leqslant 4\pi \sum_{m=n+1}^{\infty} \frac{|F'_m(z)|^2}{m+1},$$

and by letting $r \to 1^+$ and $R \to +\infty$ we get

$$\iint_{C\bar{U}} \left| \frac{\Psi(w)}{(\Psi(w)-z)^2} - \sum_{m=1}^{n} \frac{F'_m(z)}{w^{m+1}} \right|^2 d\sigma_w \leqslant 4\pi \sum_{m=n+1}^{\infty} \frac{|F'_m(z)|^2}{m+1}. \tag{13}$$

Therefore, by (12), (13) and Lemma 2.2, we conclude that $\sum_{m=1}^{\infty} a_m(f)\, F'_m(z)$ converges uniformly to f on M. This completes the proof.

Corollary 3.1. *If $P_n(z)$ is a polynomial of degree n and $a_m(P_n)$ are its generalized Faber coefficients, then $a_m(P_n) = 0$ for all $m \geqslant n+2$ and $P_n(z) = \sum_{m=1}^{n+1} a_m(P_n)\, F'_m(z)$.*

Proof. Let $z \in G$. By Theorem 3.1, we have $P_n(z) = \sum_{m=1}^{\infty} a_m(P_n)\, F'_m(z)$. It is obvious that $P_n(z)$ can be written in the form $P_n(z) = \sum_{v=1}^{n+1} A_v F'_v(z)$. Let $y(z)$ be a differentiable K-quasiconformal reflection relative to Γ. Since $y(z)$ is fixed on Γ, we get by Green's formulae

$$a_m(P_n) = -\frac{1}{\pi} \iint_{C\bar{U}} \frac{P_n(y(\Psi(w)))\, \bar{\Psi}(w)}{w^{m+1}}\, y_{\bar{\zeta}}(\Psi(w))\, d\sigma_w$$

$$= \sum_{v=1}^{n+1} -\frac{A_v}{\pi} \iint_{C\bar{U}} \frac{F'_v(y(\Psi(w)))\, \bar{\Psi}'(w)}{w^{m+1}}\, y_{\bar{\zeta}}(\Psi(w))\, d\sigma_w$$

$$= \sum_{v=1}^{n+1} -\frac{A_v}{\pi} \iint_{C\bar{U}} \frac{\partial}{\partial \bar{w}} \left(\frac{F_v(y(\Psi(w)))}{w^{m+1}} \right) d\sigma_w$$

$$= \sum_{v=1}^{n+1} \frac{A_v}{2\pi i} \int_{|w|=1} \frac{F_v(\Psi(w))}{w^{m+1}}\, dw. \tag{14}$$

Since

$$\frac{1}{2\pi i}\int_{|w|=1}\frac{F_v(\Psi(w))}{w^{m+1}}\,dw=\begin{cases}1, & \text{if} \quad v=m,\\ 0, & \text{if} \quad v\neq m,\end{cases} \tag{15}$$

see [8: p. 43], it follows that $a_m(P_n)=A_m$, for $m=1, ..., n+1$, and $a_m(P_n)=0$ for all $m\geqslant n+2$. So $P_n(z)=\sum_{m=1}^{n+1} a_m(P_n)\,F'_m(z)$.

THEOREM 3.2. *Let $\{a_m\}$ be a complex number sequence. If the series $\sum_{m=1}^{\infty} a_m F'_m(z)$ converges to a function $f\in A^2(G)$ in the norm $\|\ \|_{A^2(G)}$, then the a_m are the generalized Faber coefficients of f.*

Proof. Let $y(z)$ be a differentiable K-quasiconformal reflection relative to Γ, and $S_n(z):=\sum_{m=1}^{n+1} a_m F'_m(z)$ be the nth partial sum of $\sum_{m=1}^{\infty} a_m F'_m(z)$. Using (15), it can be shown that

$$\lim_{n\to\infty}\frac{1}{\pi}\iint_{C\bar{U}}\frac{S_n(y(\Psi(w)))\,\bar{\Psi}'(w)}{w^{m+1}}\,y_{\bar{\zeta}}(\Psi(w))\,d\sigma_w=a_m, \qquad m=1, 2, ..., \tag{16}$$

So, if m and n are natural numbers, we get by using Hölder's inequality and Lemma 2.3

$$\begin{aligned}
&|a_m(f)-a_m|\\
&\quad\leqslant\frac{1}{\pi}\left|\iint_{C\bar{U}}\frac{[f(y(\Psi(w)))-S_n(y(\Psi(w)))]\,\bar{\Psi}'(w)}{w^{m+1}}\,y_{\bar{\zeta}}(\Psi(w))\,d\sigma_w\right|\\
&\qquad+\left|-\frac{1}{\pi}\iint_{C\bar{U}}\frac{S_n(y(\Psi(w)))\,\bar{\Psi}'(w)}{w^{m+1}}\,y_{\bar{\zeta}}(\Psi(w))\,d\sigma_w-a_m\right|\\
&\quad\leqslant\frac{1}{\pi}\left(\iint_{C\bar{U}}\frac{d\sigma_w}{|w|^{2m+2}}\right)^{1/2}\\
&\qquad\times\left(\iint_{C\bar{U}}|f(y(\Psi(w)))-S_n(y(\Psi(w)))|^2\cdot|\Psi'(w)|^2\,|y_{\bar{\zeta}}(\Psi(w))|^2\,d\sigma_w\right)^{1/2}\\
&\qquad+\left|-\frac{1}{\pi}\iint_{C\bar{U}}\frac{S_n(y(\Psi(w)))\,\bar{\Psi}'(w)}{w^{m+1}}\,y_{\bar{\zeta}}(\Psi(w))\,d\sigma_w-a_m\right|\\
&\quad\leqslant\frac{1}{\sqrt{m\pi}}\left(\iint_{C\bar{G}}|((f-S_n)\circ y)(\zeta)|^2\,|y_{\bar{\zeta}}(\zeta)|^2\,d\sigma_\zeta\right)^{1/2}\\
&\qquad+\left|-\frac{1}{\pi}\iint_{C\bar{U}}\frac{S_n(y(\Psi(w)))\,\bar{\Psi}'(w)}{w^{m+1}}\,y_{\bar{\zeta}}(\Psi(w))\,d\sigma_w-a_m\right|
\end{aligned}$$

$$\leqslant \frac{\|f-S_n\|_{A^2(G)}}{\sqrt{m\pi(1-k^2)}}$$

$$+\left|-\frac{1}{\pi}\iint_{C\bar{U}} \frac{S_n(y(\Psi(w)))\,\bar{\Psi}'(w)}{w^{m+1}}\, y_{\bar{\zeta}}(\Psi(w))\, d\sigma_w - a_m\right|. \tag{17}$$

Since $\lim_{n\to\infty}\|f-S_n\|_{A^2(G)}=0$, (16) and (17) show that $a_m(f)=a_m$, and so the proof is completed.

Under the assumption of the above theorem, it is seen that the generalized Faber coefficients of the limit function are independent of the choice of the differentiable K-quasiconformal reflection.

THEOREM 3.3. *If $f\in A^2(G)$ and $S_n(f,z)=\sum_{m=1}^{n+1} a_m(f)\,F'_m(z)$ is the nth partial sum of its generalized Faber series $\sum_{m=1}^{\infty} a_m(f)\,F'_m(z)$, then*

$$\|f-S_n(f,\cdot)\|_{A^2(G)}\leqslant\sqrt{6n/(1-k^2)}\,E_n(f,G)$$

for all natural numbers n.

Proof. Let $y(z)$ be a differentiable K-quasiconformal reflection with respect to Γ, and $P_n^*(z)$ the best approximant polynomial to $f\in A^2(G)$ in the norm $\|\ \|_{A^2(G)}$. By means of Hölder's inequality, Lemma 2.3 and Corollary 3.1 we obtain

$$\begin{aligned}
&|f(z)-S_n(f,z)|\\
&\quad\leqslant|f(z)-P_n^*(z)|+|P_n^*(z)-S_n(f,z)|\\
&\quad\leqslant|f(z)-P_n^*(z)|+\left|\sum_{m=1}^{n+1}(a_m(P_n^*)-a_m(f))\,F'_m(z)\right|\\
&\quad\leqslant|f(z)-P_n^*(z)|\\
&\qquad+\frac{1}{\pi}\left|\iint_{C\bar{U}}(f\circ y-P_n^*\circ y)(\Psi(w))\,\overline{\Psi'(w)}\,y_{\bar{\zeta}}(\Psi(w))\sum_{m=1}^{n+1}\frac{F'_m(z)}{w^{m+1}}\,d\sigma_w\right|\\
&\quad\leqslant\frac{1}{\pi}\left(\iint_{C\bar{U}}|(f\circ y-P_n^*\circ y)(\Psi(w))|^2\,|\Psi'(w)|^2\,|y_{\bar{\zeta}}|^2\,d\sigma_w\right)^{1/2}\\
&\qquad\times\left(\iint_{C\bar{U}}\left|\sum_{m=1}^{n+1}\frac{F_m(z)}{w^{m+1}}\right|^2 d\sigma_w\right)^{1/2}+|f(z)-P_n^*(z)|
\end{aligned}$$

$$\leqslant |f(z) - P_n^*(z)| + \frac{1}{\pi} \left(\iint_{C\bar{G}} |(f \circ y - P_n^* \circ y)(\zeta)|^2 \, |y_{\bar{\zeta}}(\zeta)|^2 \, d\sigma_\zeta \right)^{1/2}$$

$$\times \left(\pi \sum_{m=1}^{n+1} \frac{|F_m'(z)|^2}{m} \right)^{1/2}$$

$$\leqslant |f(z) - P_n^*(z)| + \frac{1}{\sqrt{\pi(1-k^2)}} \, \|f - P_n^*\|_{A^2(G)} \left(\sum_{m=1}^{n+1} \frac{|F_m'(z)|^2}{m} \right)^{1/2}$$

$$\leqslant |f(z) - P_n^*(z)| + \frac{1}{\sqrt{\pi(1-k^2)}} \, E_n(f, G) \left(\sum_{m=1}^{n+1} \frac{|F_m'(z)|^2}{m} \right)^{1/2}$$

for all natural numbers n. This shows that

$$|f(z) - S_n(f, z)|^2 \leqslant 2 \, |f(z) - P_n^*(z)|^2 + \frac{2}{\pi(1-k^2)} \, E_n^2(f, G) \sum_{m=1}^{n+1} \frac{|F_m'(z)|^2}{m}.$$

Therefore, by integrating both sides over G and considering Lemma 2.1 we get

$$\|f - S_n(f, \cdot)\|_{A^2(G)}^2 \leqslant 2E_n^2(f, G) + \frac{2}{\pi(1-k^2)} \, E_n^2(f, G) \sum_{m=1}^{n+1} \frac{\|F_m'\|_{A^2(G)}^2}{m}$$

$$\leqslant \left(2 + \frac{2(n+1)}{1-k^2} \right) E_n^2(f, G)$$

$$\leqslant \frac{6n}{1-k^2} \, E_n^2(f, G),$$

i.e., $\|f - S_n(f, \cdot)\|_{A^2(G)} \leqslant \sqrt{6n/(1-k^2)} \, E_n(f, G)$ for all natural numbers n.

COROLLARY 3.1. *If $f \in A^2(U)$, then its generalized Faber series converges to f in the norm $\| \ \|_{A^2(U)}$.*

Proof. This is obvious from the preceding theorem and Theorem 11 and Theorem 1 in [7] and [2], respectively.

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